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Disentanglement and decoherence in a pair of qutrits under dephasing noise

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We relate disentanglement and decoherence rates in a pair of three-level atoms subjected to multi-local and collective pure dephasing noise acting in a preferred basis. The bipartite entanglement decay rate, as bounded from above by the negativity, is found to be greater than or equal to the dephasing-decoherence rates characterized by the decay of off-diagonal elements in the corresponding full density matrix describing the system or the reduced density matrix describing either qutrit, extending previous results for qubit pairs subject to such noise.

1 Introduction

Quantum entanglement can be affected by a range of noise sources, both quantum and classical in nature. Noise can give rise to a loss of entanglement over a broad range of quantum states. Recently, the relationship between dephasing-decoherence and bipartite entanglement reduction under basis-specific classical noise has been studied in two-qubit systems [1, 8, 3, 4, 5, 6, 7] and in pairs of qutrit systems [9, 10, 11]; this relationship has been studied in two-qubit systems under quantum dissipative vacuum noise as well [12]. For example, it has been noted that initially entangled two-qubit systems can suffer "entanglement sudden death," in which a two-qubit system may suddenly lose entanglement in a finite time, even though each qubit itself maintains its quantum-coherence [3, 4]. Here, we present the first general analysis of the disentanglement and dephasing of two qutrits as realized in atoms with "V"-type energy-level configuration under a classical pure dephasing noise model, at the multi-local and collective level. We compare the timescales of disentanglement and dephasing-decoherence, the latter timescale specifically in the basis on which this noise acts as in previous studies.

In Sec. 2, we introduce our dephasing model. In Sec. 3, we examine the effects of the multi-local and collective dephasing noise on a general state. Finally, in Sec. 4, we examine two specific classes of states and compare their disentanglement and decoherence rates explicitly. The bipartite-entanglement decay rate, is found to be greater than or equal to the dephasing-decoherence rates, characterized by the decay of off-diagonal elements in the corresponding density matrix describing the full system or the reduced density matrix describing either qutrit, when dephasing occurs. This result is the most general yet to be obtained in the comparative study of decoherence and disentanglement in two-qutrit systems.

2 Model

Our model consists of two-three level systems subjected to the time-dependent Hamiltonian

$$H(t) = -\frac{\mu}{2}\left[b_{\mathrm{A}}^{(1)}(t)\,\sigma_{\mathrm{z}}^{\mathrm{A}} + b_{\mathrm{B}}^{(1)}(t)\,\sigma_{\mathrm{z}}^{\mathrm{B}} + b_{\mathrm{AB}}^{(2)}(t)\left(\sigma_{\mathrm{z}}^{\mathrm{A}} + \sigma_{\mathrm{z}}^{\mathrm{B}}\right)\right], \quad (1)$$

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where $\sigma_z = \mathrm{diag}(1, e^{\frac{i2\pi}{3}}, e^{\frac{i4\pi}{3}})$ is the dephasing operator for three-level systems with subscripts denoting qutrits A, B, or both, $\hbar = 1$, the time dependent noise terms $b_X^{(i)}(i=1,2)$ refer to statistically independent classical Markov processes satisfying $\langle b_X(t)\rangle = 0$ and $\langle b_X(t)\, b_X(t')\rangle = \frac{\Gamma_1}{\mu^2}\delta(t-t')$, with $X = \mathrm{A},\mathrm{B}$; $\langle b_{\mathrm{AB}}(t)\rangle = 0$ and $\langle b_{\mathrm{AB}}(t)\, b_{\mathrm{AB}}(t')\rangle = \frac{\Gamma_2}{\mu^2}\delta(t-t')$; $\langle\cdots\rangle$ is the ensemble time average, Γ_1 and Γ_2 denote the phase-damping rates associated with $b_X(t)$ (X = A, B) and $b_{\mathrm{AB}}(t)$, respectively.

The one-qutrit standard-basis eigenstates are $\{|0\rangle, |+1\rangle, |-1\rangle\}$, for example, representing the ground state, first-excited state, and second-excited state of the atom, respectively. We assume that the states $|+1\rangle, |-1\rangle$ couple to the ground state but not to each other. Here, we notate the standard two-qutrit basis states via the obvious one-to-one correspondence $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle, |8\rangle, |9\rangle\} \doteq \{|00\rangle, |0,+1\rangle, |0,-1\rangle, |+1,0\rangle, |+1,+1\rangle, |+1,-1\rangle, |-1,0\rangle, |-1,+1\rangle, |-1,-1\rangle\}$, for simplicity.

The time-dependent density matrix for the two-qutrit system is obtained by taking ensemble averages over the three noise fields, $b_{\mathrm{A}}(t)$, $b_{\mathrm{B}}(t)$, $b_{\mathrm{AB}}(t)$, that is, $\rho(t) = \langle\rho_{\mathrm{st}}(t)\rangle_{\mathrm{A(B,AB)}}$, where the statistical density operator $\rho_{\mathrm{st}}(t)$ and the unitary operator $U(t)$ associated with $H(t)$ are $\rho_{\mathrm{st}}(t) = U(t)\,\rho(0)\,U^\dagger(t)$ and $U(t) = \exp\left[-i\int_0^t dt' H(t')\right]$, respectively. It is helpful to consider the dynamical evolution of $\rho(t)$ as a completely positive trace preserving (CPTP) linear map $\mathcal{E}(\rho)$, that is, a combination of local and collective quantum channels, any of which can be turned off in particular cases, taking an input state $\rho(0)$ to the output state $\rho(t)$ given by the operator sum $\rho(t) = \mathcal{E}(\rho(0)) = \sum_{\mu=1}^N \overline{E}_\mu^\dagger(t)\,\rho(0)\,\overline{E}_\mu(t)$, where $\overline{E}_\mu$ are decomposition operators that satisfy the completeness relation $\sum_\mu \overline{E}_\mu^\dagger \overline{E}_\mu = \mathbb{I}$. In each of the various cases considered here, the internal structure of the $\overline{E}_\mu$ in accordance with the Hamiltonian; various terms may or may not nontrivially contribute in a given case. The most general solution of $\rho(t)$, assuming that the system is not initially correlated with any of the three environments, is $\rho(t) = \sum_{i,j=1}^3 \sum_{k=1}^3 (D_k^{\mathrm{AB}\dagger} E_j^{\mathrm{B}\dagger} E_i^{\mathrm{A}\dagger})\rho(0)\left(E_i^{\mathrm{A}} E_j^{\mathrm{B}} D_k^{\mathrm{AB}}\right)$, where the terms describing the interaction with the local magnetic fields $b_{\mathrm{A}}(t)$ and $b_{\mathrm{B}}(t)$ involve the decomposition operators $E_1^{\mathrm{A}} = \mathrm{diag}(1, \gamma_{\mathrm{A}}(t), \gamma_{\mathrm{A}}(t)) \otimes \mathbb{I}_3$, $E_2^{\mathrm{A}} = \mathrm{diag}(0, \omega_{\mathrm{A}}(t), 0) \otimes \mathbb{I}_3$, $E_3^{\mathrm{A}} = \mathrm{diag}(0, 0, \omega_{\mathrm{A}}(t)) \otimes \mathbb{I}_3$, $E_1^{\mathrm{B}} = \mathbb{I}_3 \otimes \mathrm{diag}(1, \gamma_{\mathrm{B}}(t), \gamma_{\mathrm{B}}(t))$, $E_2^{\mathrm{B}} = \mathbb{I}_3 \otimes \mathrm{diag}(0, \omega_{\mathrm{B}}(t), 0)$, and $E_2^{\mathrm{B}} = \mathbb{I}_3 \otimes \mathrm{diag}(0, 0, \omega_{\mathrm{B}}(t))$, and the terms associated with the global magnetic field $b_{\mathrm{AB}}(t)$ involve the operators $D_1^{\mathrm{AB}} = \mathrm{diag}(\gamma_{\mathrm{AB}}(t), 1, 1, 1, \gamma_{\mathrm{AB}}(t), 1, 1, 1, \gamma_{\mathrm{AB}}(t))$, $D_2^{\mathrm{AB}} = \mathrm{diag}(\omega_{\mathrm{AB1}}(t), 0, 0, 0, \omega_{\mathrm{AB2}}(t), 0, 0, 0, \omega_{\mathrm{AB2}}(t))$, $D_3^{\mathrm{AB}} = \mathrm{diag}(0, 0, 0, 0, \omega_{\mathrm{AB3}}(t), 0, 0, 0, \omega_{\mathrm{AB3}}(t))$. The time-dependent parameters appearing in the matrices above are given by: $\gamma_i(t) = e^{-t/2T_i}$, $\omega_i(t) = \sqrt{1-\gamma_i^2}$, $\gamma(t) = e^{-t/2T_i}$, $\omega_{i1}(t) = \sqrt{1-\gamma_i^2}$, $\omega_{i2}(t) = -\gamma_i^2\sqrt{1-\gamma_i^2}$, and $\omega_{i3}(t) = \sqrt{\left(1-\gamma_i^2\right)\left(1-\gamma_i^4\right)}$, where $T_i = \frac{1}{\Gamma_i}(i=1,2)$ are the phase-relaxation times associated with the the pertinent qubits A and B, respectively, as introduced in [8], with Γ_i being the rate parameters. From here on, for tractability of notation, time does not explicitly appear as an argument for these quantities but is implied.

Dephasing-decoherence rates are characterized by the decay of off-diagonal elements in the corresponding full density matrix describing the system or the reduced density matrix describing either qutrit given via characteristic decay times. Entanglement is bounded from above by the negativity $\mathcal{N}(\rho) = \frac{\|\rho^{T_{\mathrm{A}}}\|_1 - 1}{2}$, where $\rho^{T_{\mathrm{A}}}$ is the partial transpose of the density matrix with respect to qutrit A and $\|\cdot\|_1$ denotes the trace norm [13].

3 General Case

In the standard-basis representation of Eq. 2, the generic pure state of the two-qutrit system is $|\Psi\rangle_{\mathrm{AB}} = \bar{a}_1|1\rangle + \bar{a}_2|2\rangle + \bar{a}_3|3\rangle + \bar{a}_4|4\rangle + \bar{a}_5|5\rangle + \bar{a}_6|6\rangle + \bar{a}_7|7\rangle + \bar{a}_8|8\rangle + \bar{a}_9|9\rangle$, a normalized state-vector with $\bar{a}_i \in \mathbb{C}$ and $\sum_{i=1}^9 \bar{a}_i^2 = 1$. Our analysis proceeds as follows. We find the explicit time evolution of the general state subjected to noise from the multi-local dephasing channel $\mathcal{EF}$ and the collective dephasing channel $\mathcal{D}$. The decoherence timescales are then determined, as characterized by the decay of the off-diagonal elements at the level of the full density matrix of two-qutrits, as well as the reduced density matrix of each individual qutrit, for each of the dephasing cases, multi-local and collective. We then analyze disentanglement timescales, using the monotone of negativity, when there is decoherence for the general

state in each of these cases. In order to compare decoherence and disentanglement behavior explicitly, we then consider the the behavior of two particular subclasses of states, the *robust* class and the *fragile* class.

First, consider the most general initial two-qutrit pure state, $\rho_{\mathrm{AB}}(0) = P(|\Psi\rangle_{\mathrm{AB}})$, where $P(|\Psi\rangle_{\mathrm{AB}})$ is the projector corresponding to the generic state-vector argument $|\Psi\rangle_{\mathrm{AB}}$ explicitly given above, under multi-local and collective dephasing noise.

3.1 *General Case: Multi-Local Dephasing Channel $\mathcal{E}\mathcal{F}$*

$$\rho_{\mathrm{AB}}^{\mathrm{G},\mathcal{E}F}(t) = \begin{pmatrix}
|\bar{a}_1|^2 & \bar{a}_1\bar{a}_2^*\gamma_{\mathrm{B}} & \bar{a}_1\bar{a}_3^*\gamma_{\mathrm{B}} & \bar{a}_1\bar{a}_4^*\gamma_{\mathrm{A}} & \bar{a}_1\bar{a}_5^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_1\bar{a}_6^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_1\bar{a}_7^*\gamma_{\mathrm{A}} & \bar{a}_1\bar{a}_8^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_1\bar{a}_9^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} \\
\bar{a}_2\bar{a}_1^*\gamma_{\mathrm{B}} & |\bar{a}_2|^2 & \bar{a}_2\bar{a}_3^*\gamma_{\mathrm{B}}^2 & \bar{a}_2\bar{a}_4^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_2\bar{a}_5^*\gamma_{\mathrm{A}} & \bar{a}_2\bar{a}_6^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 & \bar{a}_2\bar{a}_7^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_2\bar{a}_8^*\gamma_{\mathrm{A}} & \bar{a}_2\bar{a}_9^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 \\
\bar{a}_3\bar{a}_1^*\gamma_{\mathrm{B}} & \bar{a}_3\bar{a}_2^*\gamma_{\mathrm{B}}^2 & |\bar{a}_3|^2 & \bar{a}_3\bar{a}_4^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_3\bar{a}_5^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 & \bar{a}_3\bar{a}_6^*\gamma_{\mathrm{A}} & \bar{a}_3\bar{a}_7^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_3\bar{a}_8^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 & \bar{a}_3\bar{a}_9^*\gamma_{\mathrm{A}} \\
\bar{a}_4\bar{a}_1^*\gamma_{\mathrm{A}} & \bar{a}_4\bar{a}_2^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_4\bar{a}_3^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & |\bar{a}_4|^2 & \bar{a}_4\bar{a}_5^*\gamma_{\mathrm{B}} & \bar{a}_4\bar{a}_6^*\gamma_{\mathrm{B}} & \bar{a}_4\bar{a}_7^*\gamma_{\mathrm{A}}^2 & \bar{a}_4\bar{a}_8^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} & \bar{a}_4\bar{a}_9^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} \\
\bar{a}_5\bar{a}_1^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_5\bar{a}_2^*\gamma_{\mathrm{A}} & \bar{a}_5\bar{a}_3^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 & \bar{a}_5\bar{a}_4^*\gamma_{\mathrm{B}} & |\bar{a}_5|^2 & \bar{a}_5\bar{a}_6^*\gamma_{\mathrm{B}}^2 & \bar{a}_5\bar{a}_7^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} & \bar{a}_5\bar{a}_8^*\gamma_{\mathrm{A}}^2 & \bar{a}_5\bar{a}_9^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}}^2 \\
\bar{a}_6\bar{a}_1^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_6\bar{a}_2^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 & \bar{a}_6\bar{a}_3^*\gamma_{\mathrm{A}} & \bar{a}_6\bar{a}_4^*\gamma_{\mathrm{B}} & \bar{a}_6\bar{a}_5^*\gamma_{\mathrm{B}}^2 & |\bar{a}_6|^2 & \bar{a}_6\bar{a}_7^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} & \bar{a}_6\bar{a}_8^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}}^2 & \bar{a}_6\bar{a}_9^*\gamma_{\mathrm{A}}^2 \\
\bar{a}_7\bar{a}_1^*\gamma_{\mathrm{A}} & \bar{a}_7\bar{a}_2^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_7\bar{a}_3^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_7\bar{a}_4^*\gamma_{\mathrm{A}}^2 & \bar{a}_7\bar{a}_5^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} & \bar{a}_7\bar{a}_6^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} & |\bar{a}_7|^2 & \bar{a}_7\bar{a}_8^*\gamma_{\mathrm{B}} & \bar{a}_7\bar{a}_9^*\gamma_{\mathrm{B}} \\
\bar{a}_8\bar{a}_1^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_8\bar{a}_2^*\gamma_{\mathrm{A}} & \bar{a}_8\bar{a}_3^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 & \bar{a}_8\bar{a}_4^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} & \bar{a}_8\bar{a}_5^*\gamma_{\mathrm{A}}^2 & \bar{a}_8\bar{a}_6^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}}^2 & \bar{a}_8\bar{a}_7^*\gamma_{\mathrm{B}} & |\bar{a}_8|^2 & \bar{a}_8\bar{a}_9^*\gamma_{\mathrm{B}}^2 \\
\bar{a}_9\bar{a}_1^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \bar{a}_9\bar{a}_2^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}}^2 & \bar{a}_9\bar{a}_3^*\gamma_{\mathrm{A}} & \bar{a}_9\bar{a}_4^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}} & \bar{a}_9\bar{a}_5^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}}^2 & \bar{a}_9\bar{a}_6^*\gamma_{\mathrm{A}}^2 & \bar{a}_9\bar{a}_7^*\gamma_{\mathrm{B}} & \bar{a}_9\bar{a}_8^*\gamma_{\mathrm{B}}^2 & |\bar{a}_9|^2
\end{pmatrix} \quad (2)$$

is the time-evolved full density matrix for multi-local dephasing. Our density matrices are labeled from here on as $\rho_{\mathrm{X}}^{\mathrm{c},\mathcal{C}}$, with X denoting the pertinent qubits (A, B, AB), c is the class examined (G,F,R) as described in the next section, and $\mathcal{C}$ denoting the pertinent dephasing channel ($\mathcal{E}, \mathcal{F}, \mathcal{D}$) As noted above, for tractability of notation, time does not explicitly appear as an argument for the γ_A and γ_B. Note that one can recover the local dephasing channel $\mathcal{E}(\mathcal{F})$ for qutrit A(B) by effectively freezing the time parameter in the exponential of the $\gamma_B(\gamma_A)$ factors of the opposite channel. The effect of two local dephasing noise sources, one for each qutrit, is simply the combination of independent effects of the individual local dephasing channel just described. Thus, the combined matrix is just the component-wise multiplication of exponential decay factors arising from each of the local dephasing channels. Due to their appearance in exponents, the decay rates add.

Let us designate the different timescales as $\tau_{i-\mathrm{dec},\mathcal{C}}^{c(j)}$ and $\tau_{i-\mathrm{dis},\mathcal{C}}^{c(j)}$, representing the decoherence and disentanglement timescales, respectively; i denotes the number of qutrits affected ($i = 1, 2$), $\mathcal{C}$ denotes the pertinent dephasing channel ($\mathcal{E}, \mathcal{F}, \mathcal{D}$), j is used as an index further to discriminate the timescales, and c denotes the class examined (G,F,R). The differing timescales of reduction of off-diagonal elements consist of various combinations of γ_A and γ_B. Here, we have the four timescales of decay as $\tau_{2-\mathrm{dec},\mathcal{E}\mathcal{F}}^{G(1)} = 2(\frac{1}{\Gamma_1})$, $\tau_{2-\mathrm{dec},\mathcal{E}\mathcal{F}}^{G(2)} = (\frac{1}{\Gamma_1})$, $\tau_{2-\mathrm{dec},\mathcal{E}\mathcal{F}}^{G(3)} = 2(\frac{1}{3\Gamma_1})$, and $\tau_{2-\mathrm{dec},\mathcal{E}\mathcal{F}}^{G(4)} = (\frac{1}{2\Gamma_1})$. The reduced density matrices of the individual qutrit subsystems are

$$\rho_{\mathrm{A}}^{\mathrm{G},\mathcal{E}F}(t) = \mathrm{Tr}_{\mathrm{B}}\rho_{\mathrm{AB}}^{\mathrm{G},\mathcal{E}F}(t) = \begin{pmatrix}
|\bar{a}_1|^2 + |\bar{a}_2|^2 + |\bar{a}_3|^2 & (\bar{a}_1\bar{a}_4^* + \bar{a}_2\bar{a}_5^* + \bar{a}_3\bar{a}_6^*)\gamma_{\mathrm{A}} & (\bar{a}_1\bar{a}_7^* + \bar{a}_2\bar{a}_8^* + \bar{a}_3\bar{a}_9^*)\gamma_{\mathrm{A}}, \\
(\bar{a}_4\bar{a}_1^* + \bar{a}_5\bar{a}_2^* + \bar{a}_6\bar{a}_3^*)\gamma_{\mathrm{A}} & |\bar{a}_4|^2 + |\bar{a}_5|^2 + |\bar{a}_6|^2 & (\bar{a}_4\bar{a}_7^* + \bar{a}_5\bar{a}_8^* + \bar{a}_6\bar{a}_9^*)\gamma_{\mathrm{A}}^2 \\
(\bar{a}_7\bar{a}_1^* + \bar{a}_8\bar{a}_2^* + \bar{a}_9\bar{a}_3^*)\gamma_{\mathrm{A}} & (\bar{a}_7\bar{a}_4^* + \bar{a}_8\bar{a}_5^* + \bar{a}_9\bar{a}_6^*)\gamma_{\mathrm{A}}^2 & |\bar{a}_7|^2 + |\bar{a}_8|^2 + |\bar{a}_9|^2
\end{pmatrix} \quad (3)$$

$$\rho_{\mathrm{B}}^{\mathrm{G},\mathcal{E}F}(t) = \mathrm{Tr}_{\mathrm{A}}\rho_{\mathrm{AB}}^{\mathrm{G},\mathcal{E}F}(t) = \begin{pmatrix}
|\bar{a}_1|^2 + |\bar{a}_4|^2 + |\bar{a}_7|^2 & (\bar{a}_1\bar{a}_2^* + \bar{a}_4\bar{a}_5^* + \bar{a}_7\bar{a}_8^*)\gamma_{\mathrm{B}} & (\bar{a}_1\bar{a}_3^* + \bar{a}_4\bar{a}_6^* + \bar{a}_7\bar{a}_9^*)\gamma_{\mathrm{B}} \\
(\bar{a}_2\bar{a}_1^* + \bar{a}_5\bar{a}_4^* + \bar{a}_8\bar{a}_7^*)\gamma_{\mathrm{B}} & |\bar{a}_2|^2 + |\bar{a}_5|^2 + |\bar{a}_8|^2 & (\bar{a}_2\bar{a}_3^* + \bar{a}_5\bar{a}_5^* + \bar{a}_8\bar{a}_9^*)\gamma_{\mathrm{B}}^2 \\
(\bar{a}_3\bar{a}_1^* + \bar{a}_6\bar{a}_4^* + \bar{a}_9\bar{a}_7^*)\gamma_{\mathrm{B}} & (\bar{a}_3\bar{a}_2^* + \bar{a}_6\bar{a}_5^* + \bar{a}_9\bar{a}_8^*)\gamma_{\mathrm{B}}^2 & |\bar{a}_3|^2 + |\bar{a}_6|^2 + |\bar{a}_9|^2
\end{pmatrix} \quad (4)$$

Again, one finds differing dephasing decoherence timescales. In particular, the off-diagonal elements decay in two different timescales before the density matrix becomes fully diagonal in the basis under consideration, $\tau_{1-\mathrm{dec},\mathcal{E}\mathcal{F}}^{G(1)} = 2(\frac{1}{\Gamma_1})$ and $\tau_{1-\mathrm{dec},\mathcal{E}\mathcal{F}}^{G(2)} = (\frac{1}{\Gamma_1})$.

The negativity, which bounds the (non-bound) entanglement,

$$\begin{aligned}
\mathcal{N}\left[(\rho^{\mathcal{EF}}_{\mathrm{AB}}(t))^{\mathrm{T_A}}\right] = \frac{1}{2}\Bigg(& -1+\sqrt{\left[|\bar{a}_1|^2+(|\bar{a}_4|^2+|\bar{a}_7|^2)\gamma_{\mathrm{A}}^2\right]\left[|\bar{a}_1|^2+(|\bar{a}_2|^2+|\bar{a}_3|^2)\gamma_{\mathrm{B}}^2\right]} \\
& +\sqrt{\left[|\bar{a}_7|^2+|\bar{a}_1|^2\,\gamma_{\mathrm{A}}^2+|\bar{a}_4|^2\,\gamma_{\mathrm{A}}^4\right]\left[|\bar{a}_7|^2+(|\bar{a}_8|^2+|\bar{a}_9|^2)\gamma_{\mathrm{B}}^2\right]} \\
& +\sqrt{\left[|\bar{a}_4|^2+|\bar{a}_1|^2\,\gamma_{\mathrm{A}}^2+|\bar{a}_7|^2\,\gamma_{\mathrm{A}}^4\right]\left[|\bar{a}_4|^2+(|\bar{a}_5|^2+|\bar{a}_6|^2)\gamma_{\mathrm{B}}^2\right]} \\
& +\sqrt{\left[|\bar{a}_3|^2+(|\bar{a}_6|^2+|\bar{a}_9|^2)\gamma_{\mathrm{A}}^2\right]\left[|\bar{a}_3|^2+|\bar{a}_1|^2\,\gamma_{\mathrm{B}}^2+|\bar{a}_2|^2\,\gamma_{\mathrm{B}}^4\right]} \\
& +\sqrt{\left[|\bar{a}_6|^2+|\bar{a}_3|^2\,\gamma_{\mathrm{A}}^2+|\bar{a}_9|^2\,\gamma_{\mathrm{A}}^4\right]\left[|\bar{a}_6|^2+|\bar{a}_4|^2\,\gamma_{\mathrm{B}}^2+|\bar{a}_5|^2\,\gamma_{\mathrm{B}}^4\right]} \\
& +\sqrt{\left[|\bar{a}_9|^2+|\bar{a}_3|^2\,\gamma_{\mathrm{A}}^2+|\bar{a}_6|^2\,\gamma_{\mathrm{A}}^4\right]\left[|\bar{a}_9|^2+|\bar{a}_7|^2\,\gamma_{\mathrm{B}}^2+|\bar{a}_8|^2\,\gamma_{\mathrm{B}}^4\right]} \\
& +\sqrt{\left[|\bar{a}_2|^2+(|\bar{a}_5|^2+|\bar{a}_8|^2)\gamma_{\mathrm{A}}^2\right]\left[|\bar{a}_2|^2+|\bar{a}_1|^2\,\gamma_{\mathrm{B}}^2+|\bar{a}_3|^2\,\gamma_{\mathrm{B}}^4\right]} \\
& +\sqrt{\left[|\bar{a}_8|^2+|\bar{a}_2|^2\,\gamma_{\mathrm{A}}^2+|\bar{a}_5|^2\,\gamma_{\mathrm{A}}^4\right]\left[|\bar{a}_8|^2+|\bar{a}_7|^2\,\gamma_{\mathrm{B}}^2+|\bar{a}_9|^2\,\gamma_{\mathrm{B}}^4\right]} \\
& +\sqrt{\left[|\bar{a}_5|^2+|\bar{a}_2|^2\,\gamma_{\mathrm{A}}^2+|\bar{a}_8|^2\,\gamma_{\mathrm{A}}^4\right]\left[|\bar{a}_5|^2+|\bar{a}_4|^2\,\gamma_{\mathrm{B}}^2+|\bar{a}_6|^2\,\gamma_{\mathrm{B}}^4\right]}\ \Bigg)\,, \qquad (5)
\end{aligned}$$

is rather unwieldy. It is therefore illuminating to characterize its behavior for each of the dephasing cases: no dephasing, one-qutrit local dephasing, and two-qutrit multi-local dephasing. In the expression for negativity, we can isolate the behavior of one local dephasing channel with respect to the other by setting the decay factors corresponding to the other channel to one, effectively freezing that factor to time zero.

(1) In cases where there is no dephasing, any decay factors multiplying probability amplitudes do not fall off exponentially but instead their value remains at unity. $\mathcal{N}(\rho(0)) = \mathcal{N}(\rho(t)) = 1$, which corresponds to a maximally entangled state for all time. Thus the (trivial) dephasing and decoherence rates are equal.

(2) In the case of a single local dephasing channel, either channel $\mathcal{E}$ on qutrit A only or channel $\mathcal{F}$ on qutrit B only, the negativity $\mathcal{N}(\rho(t)) \to \mathrm{c} \geq 0$ in the large-time limit.

(3) Finally, in the case in which there is dephasing on both local channels $\mathcal{E}$ and $\mathcal{F}$, $\mathcal{N}(\rho(t)) \to 0$ in the large-time limit.

One then notes that in all these cases decoherence never proceeds faster than disentanglement, as we previously found to be the case for qubit pairs [7]. In later sections, when examining specific classes of states, we compare decoherence and disentanglement more explicitly.

3.2 *General Case: Collective Dephasing Channel $\mathcal{D}$*

When subjected to collective dephasing noise, the time-evolved density matrix of the two-qutrit system in general case is given by

$$\rho_{\rm AB}^{\rm G,\mathcal{D}}(t)=\begin{pmatrix}
|\bar a_1|^2 & \bar a_1\bar a_2^*\gamma & \bar a_1\bar a_3^*\gamma & \bar a_1\bar a_4^*\gamma & \bar a_1\bar a_5^*\gamma^4 & \bar a_1\bar a_6^*\gamma & \bar a_1\bar a_7^*\gamma & \bar a_1\bar a_8^*\gamma & \bar a_1\bar a_9^*\gamma^4\\
\bar a_2\bar a_1^*\gamma & |\bar a_2|^2 & \bar a_2\bar a_3^* & \bar a_2\bar a_4^* & \bar a_2\bar a_5^*\gamma & \bar a_2\bar a_6^* & \bar a_2\bar a_7^* & \bar a_2\bar a_8^* & \bar a_2\bar a_9^*\gamma\\
\bar a_3\bar a_1^*\gamma & \bar a_3\bar a_2^* & |\bar a_3|^2 & \bar a_3\bar a_4^* & \bar a_3\bar a_5^*\gamma & \bar a_3\bar a_6^* & \bar a_3\bar a_7^* & \bar a_3\bar a_8^* & \bar a_3\bar a_9^*\gamma\\
\bar a_4\bar a_1^*\gamma & \bar a_4\bar a_2^* & \bar a_4\bar a_3^* & |\bar a_4|^2 & \bar a_4\bar a_5^*\gamma & \bar a_4\bar a_6^* & \bar a_4\bar a_7^* & \bar a_4\bar a_8^* & \bar a_4\bar a_9^*\gamma\\
\bar a_5\bar a_1^*\gamma^4 & \bar a_5\bar a_2^*\gamma & \bar a_5\bar a_3^*\gamma & \bar a_5\bar a_4^*\gamma & |\bar a_5|^2 & \bar a_5\bar a_6^*\gamma & \bar a_5\bar a_7^*\gamma & \bar a_5\bar a_8^*\gamma & \bar a_5\bar a_9^*\\
\bar a_6\bar a_1^*\gamma & \bar a_6\bar a_2^* & \bar a_6\bar a_3^* & \bar a_6\bar a_4^* & \bar a_6\bar a_5^*\gamma & |\bar a_6|^2 & \bar a_6\bar a_7^* & \bar a_6\bar a_8^* & \bar a_6\bar a_9^*\gamma\\
\bar a_7\bar a_1^*\gamma & \bar a_7\bar a_2^* & \bar a_7\bar a_3^* & \bar a_7\bar a_4^* & \bar a_7\bar a_5^*\gamma & \bar a_7\bar a_6^* & |\bar a_7|^2 & \bar a_7\bar a_8^* & \bar a_7\bar a_9^*\gamma\\
\bar a_8\bar a_1^*\gamma & \bar a_8\bar a_2^* & \bar a_8\bar a_3^* & \bar a_8\bar a_4^* & \bar a_8\bar a_5^*\gamma & \bar a_8\bar a_6^* & \bar a_8\bar a_7^* & |\bar a_8|^2 & \bar a_8\bar a_9^*\gamma\\
\bar a_9\bar a_1^*\gamma^4 & \bar a_9\bar a_2^*\gamma & \bar a_9\bar a_3^*\gamma & \bar a_9\bar a_4^*\gamma & \bar a_9\bar a_5^* & \bar a_9\bar a_6^*\gamma & \bar a_9\bar a_7^*\gamma & \bar a_9\bar a_8^*\gamma & |\bar a_9|^2
\end{pmatrix}. \tag{6}$$

Note the form of the matrix, in which there are regions unaffected by dephasing, with decoherence effects along the edges. We show in the next section that this gives rise to a class of states that can exist in these decoherence free subspaces, where entanglement is also be explicitly characterized. We see in the above general matrix that there exist two timescales of off-diagonal element decay, $\tau^{G(1)}_{2-{\rm dec},\mathcal{D}}=2(\frac{1}{\Gamma_2})$ and $\tau^{G(2)}_{2-{\rm dec},\mathcal{D}}=(\frac{1}{2\Gamma_2})$.

The reduced density matrices are given by the following matrices,

$$\rho_{\rm A}^{\rm G,\mathcal{D}}(t)={\rm Tr_B}\rho_{\rm AB}^{\rm G,\mathcal{D}}(t)=\begin{pmatrix}
|\bar a_1|^2+|\bar a_2|^2+|\bar a_3|^2 & \bar a_3\bar a_6^*+\bar a_1\bar a_4^*\gamma+\bar a_2\bar a_5^*\gamma & \bar a_2\bar a_8^*+\bar a_1\bar a_7^*\gamma+\bar a_3\bar a_9^*\gamma\\
\bar a_6\bar a_3^*+\bar a_4\bar a_1^*\gamma+\bar a_5\bar a_2^*\gamma & |\bar a_4|^2+|\bar a_5|^2+|\bar a_6|^2 & \bar a_4\bar a_7^*+\bar a_5\bar a_8^*\gamma+\bar a_6\bar a_9^*\gamma\\
\bar a_8\bar a_2^*+\bar a_7\bar a_1^*\gamma+\bar a_9\bar a_3^*\gamma & \bar a_7\bar a_4^*+\bar a_8\bar a_5^*\gamma+\bar a_9\bar a_6^*\gamma & |\bar a_7|^2+|\bar a_8|^2+|\bar a_9|^2
\end{pmatrix} \tag{7}$$

$$\rho_{\rm B}^{\rm G,\mathcal{D}}(t)={\rm Tr_A}\rho_{\rm AB}^{\rm G,\mathcal{D}}(t)=\begin{pmatrix}
|\bar a_1|^2+|\bar a_4|^2+|\bar a_7|^2 & \bar a_7\bar a_8^*+\bar a_1\bar a_2^*\gamma+\bar a_4\bar a_5^*\gamma & \bar a_4\bar a_6^*+\bar a_1\bar a_3^*\gamma+\bar a_7\bar a_9^*\gamma\\
\bar a_8\bar a_7^*+\bar a_2\bar a_1^*\gamma+\bar a_5\bar a_4^*\gamma & |\bar a_2|^2+|\bar a_5|^2+|\bar a_8|^2 & \bar a_2\bar a_3^*+\bar a_5\bar a_6^*\gamma+\bar a_8\bar a_9^*\gamma\\
\bar a_6\bar a_4^*+\bar a_3\bar a_1^*\gamma+\bar a_9\bar a_7^*\gamma & \bar a_3\bar a_2^*+\bar a_5\bar a_5^*\gamma+\bar a_9\bar a_8^*\gamma & |\bar a_3|^2+|\bar a_6|^2+|\bar a_9|^2
\end{pmatrix}. \tag{8}$$

Note that the single-qutrit reduced density matrices do not become entirely diagonal. The reason, as we stated above, is the existence of decoherence free subspaces. In this case, there exists only one timescale in which a subset of off-diagonal elements decay, $\tau^{G(1)}_{2-{\rm dec},\mathcal{D}}=2(\frac{1}{\Gamma_2})$.

The expression for negativity for the general class under the collective dephasing channel, is similar in character to the negativity found in the previous subsection for the multi-local dephasing channel given by Eq. 5. Therefore, for clarity, we defer our discussion of the behavior of the negativity in the case of the collective dephasing channel until after treating specific states to the next section.

4 Specific Classes

In order better to distinguish decoherence and disentanglement behavior, let us now turn our attention to two specific classes of states. In the standard-basis representation, the generic pure state of a two-qutrit system is $|\Psi\rangle=\bar a_1|1\rangle+\bar a_2|2\rangle+\bar a_3|3\rangle+\bar a_4|4\rangle+\bar a_5|5\rangle+\bar a_6|6\rangle+\bar a_7|7\rangle+\bar a_8|8\rangle+\bar a_9|9\rangle$, a normalized state vector with $\bar a_i\in\mathbb{C}$. The generic class of two-qutrit pure states represented by $|\Psi\rangle$ contains two subclasses of interest, distinguished by their coherence behavior in large timescales under collective dephasing noise, that is, dephasing in which each qutrit interacts with the same collective noise $b_{\rm AB}(t)$. One class is seen to be *fragile*, whereas the other is *robust*.

(i) The *fragile* class $|\phi\rangle=\bar a_1|1\rangle+\bar a_5|5\rangle+\bar a_9|9\rangle$, in which $\bar a_1,\bar a_5,$ and $\bar a_9$ may be non-zero and all other terms $\bar a_i=0$, has the forms

$$|\phi_1\rangle = \bar{a}_1|1\rangle + \bar{a}_5|5\rangle \ , \tag{9}$$

$$|\phi_2\rangle = \bar{a}_1|1\rangle + \bar{a}_9|9\rangle \ , \tag{10}$$

$$|\phi_3\rangle = \bar{a}_5|5\rangle + \bar{a}_9|9\rangle \ . \tag{11}$$

(ii) The *robust* class $|\psi\rangle = \bar{a}_2|2\rangle + \bar{a}_3|3\rangle + \bar{a}_4|4\rangle + \bar{a}_6|6\rangle + \bar{a}_7|7\rangle + \bar{a}_8|8\rangle$, in which all $\bar{a}_i$ listed may be non-zero and $\bar{a}_1 = \bar{a}_9 = 0$, has the forms

$$|\psi_1\rangle = \bar{a}_2|2\rangle + \bar{a}_4|4\rangle \ , \tag{12}$$

$$|\psi_2\rangle = \bar{a}_3|3\rangle + \bar{a}_7|7\rangle \ , \tag{13}$$

$$|\psi_3\rangle = \bar{a}_6|6\rangle + \bar{a}_8|8\rangle \ . \tag{14}$$

The specific forms of the two classes above constitute Bell-like states in the Hilbert space of our two-qutrit system, similarly to the qubit-pair-state classification given in [8].

4.1 *Fragile Class*

The initial density matrix representing the two-qutrit fragile class is given by

$$\rho^{\mathrm{F}}_{\mathrm{AB}}(0) = P(|\phi\rangle) \equiv \begin{pmatrix} |\bar{a}_1|^2 & \cdots & \bar{a}_1\bar{a}_5^* & \cdots & \bar{a}_1\bar{a}_9^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \bar{a}_5\bar{a}_1^* & \cdots & |\bar{a}_5|^2 & \cdots & \bar{a}_5\bar{a}_9^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \bar{a}_9\bar{a}_1^* & \cdots & \bar{a}_9\bar{a}_5^* & \cdots & |\bar{a}_9|^2 \end{pmatrix} , \tag{15}$$

where the dots denote the remainder of the density matrix, which is filled out by zero entries. Let us turn our attention to the behavior of this state when subjected to the multi-local dephasing $\mathcal{EF}$ acting on qutrits A and B. Recall that the multi-local dephasing channel captures the features of each of the local dephasing channels fully, so for this analysis, we consider the multi-local dephasing channel only, which provides information about each of the individual dephasing channels if one simply turns off one of the channels.

4.1.1 *Fragile Class: Multi-Local Dephasing Channel $\mathcal{EF}$.*

The density matrix subjected to noise is

$$\rho^{\mathrm{F},\mathcal{EF}}_{\mathrm{AB}}(t) = \begin{pmatrix} |\bar{a}_1|^2 & \cdots & \bar{a}_1\bar{a}_5^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \cdots & \bar{a}_1\bar{a}_9^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \bar{a}_5\bar{a}_1^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \cdots & |\bar{a}_5|^2 & \cdots & \bar{a}_5\bar{a}_9^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}}^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \bar{a}_9\bar{a}_1^*\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} & \cdots & \bar{a}_9\bar{a}_5^*\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}}^2 & \cdots & |\bar{a}_9|^2 \end{pmatrix} , \tag{16}$$

which has two different decoherence times of $\tau^{\mathrm{F}}_{2-\mathrm{dec},\mathcal{EF}} = (\frac{1}{\Gamma_1})$ and $\tau^{\mathrm{F}}_{2-\mathrm{dec},\mathcal{EF}} = (\frac{1}{2\Gamma_1})$. By comparison, the disentanglement timescales deriving from the negativity, $\mathcal{N}\left(\rho^{\mathcal{EF}}_{\mathrm{AB}}(t)\right) = \frac{\|\rho^{\mathrm{T_A}}\|_1 - 1}{2} = \frac{\|\rho^{\mathrm{T_B}}\|_1 - 1}{2} = \left(|\bar{a}_1|\,|\bar{a}_5| + |\bar{a}_1|\,|\bar{a}_9|\right)\gamma_{\mathrm{A}}\gamma_{\mathrm{B}} + |\bar{a}_5|\,|\bar{a}_9|\,\gamma_{\mathrm{A}}^2\gamma_{\mathrm{B}}^2$ are $\tau^{\mathrm{F}(1)}_{\mathrm{dis},\mathcal{EF}} = (\frac{1}{\Gamma_1})$ and $\tau^{\mathrm{F}(2)}_{\mathrm{dis},\mathcal{EF}} = (\frac{1}{2\Gamma_1})$.

The reduced density matrices of qutrit A and B are

$$\rho_{\text{reduced}}\left(t\right)=\text{Tr}_{\text{B}}\rho_{\text{AB}}^{\text{F},\mathcal{EF}}\left(t\right)=\text{Tr}_{\text{A}}\rho_{\text{AB}}^{\text{F},\mathcal{EF}}\left(t\right)=\begin{pmatrix}\left|\bar{a}_1\right|^2 & 0 & 0\\ 0 & \left|\bar{a}_5\right|^2 & 0\\ 0 & 0 & \left|\bar{a}_9\right|^2\end{pmatrix}. \tag{17}$$

We see that they are both initially fully mixed, so that no further decoherence is possible.

Comparing timescales we see that decoherence never proceeds faster than disentanglement, $\tau^{\text{F}}_{\text{dis},\mathcal{EF}}\leq\tau^{\text{F}}_{2-\text{dec},\mathcal{EF}}$ and $\tau^{\text{F}}_{\text{dis},\mathcal{EF}}\leq\tau^{\text{F}}_{1-\text{dec},\mathcal{EF}}$

4.1.2 ***Fragile Class: Collective Dephasing Channel $\mathcal{D}$.***

The two-qutrit density matrix, as given by

$$\rho_{\text{AB}}^{\text{F},\mathcal{D}}\left(t\right)=\begin{pmatrix}\left|\bar{a}_1\right|^2 & \cdots & \bar{a}_1\bar{a}_5^*\gamma^4 & \cdots & \bar{a}_1\bar{a}_9^*\gamma^4\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \bar{a}_5\bar{a}_1^*\gamma^4 & \cdots & \left|\bar{a}_5\right|^2 & \cdots & \bar{a}_5\bar{a}_9^*\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \bar{a}_9\bar{a}_1^*\gamma^4 & \cdots & \bar{a}_9\bar{a}_5^* & \cdots & \left|\bar{a}_9\right|^2\end{pmatrix}, \tag{18}$$

decays according to a single timescale, $\tau^{\text{F}}_{2-\text{dec},\mathcal{D}}=(\frac{1}{2\Gamma})$. The disentanglement, characterized via the negativity, $\mathcal{N}\left(\rho_{\text{AB}}^{\mathcal{D}}\left(t\right)\right)=\frac{\|\rho^{\text{T}_\text{A}}\|_1-1}{2}=\frac{\|\rho^{\text{T}_\text{B}}\|_1-1}{2}=\left|\bar{a}_5\right|\left|\bar{a}_9\right|+\left(\left|\bar{a}_1\right|\left|\bar{a}_5\right|+\left|\bar{a}_1\right|\left|\bar{a}_9\right|\right)\gamma^4$. Disentanglement proceeds at only a single timescale, $\tau^{\text{F}}_{\text{dis},\mathcal{D}}=(\frac{1}{2\Gamma})$.

The single qutrit matrices are always

$$\rho_{\text{reduced}}\left(t\right)=\text{Tr}_{\text{B}}\rho_{\text{AB}}^{\text{F},\mathcal{D}}\left(t\right)=\text{Tr}_{\text{A}}\rho_{\text{AB}}^{\text{F},\mathcal{D}}\left(t\right)=\begin{pmatrix}\left|\bar{a}_1\right|^2 & 0 & 0\\ 0 & \left|\bar{a}_5\right|^2 & 0\\ 0 & 0 & \left|\bar{a}_9\right|^2\end{pmatrix}. \tag{19}$$

Again, we see that the single qutrit states are both fully mixed, so that no further decoherence is possible. Comparing timescales, we see that decoherence never proceeds faster than disentanglement: $\tau^{\text{F}}_{\text{dis},\mathcal{D}}\leq\tau^{\text{F}}_{2-\text{dec},\mathcal{D}}$.

4.2 ***Robust Class***

The two-qutrit density matrix for the robust class, under the multi-local dephasing channel, is given by

$$\rho_{\text{AB}}^{\text{R}}\left(0\right)=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \left|\bar{a}_2\right|^2 & \bar{a}_2\bar{a}_3^* & \bar{a}_2\bar{a}_4^* & 0 & \bar{a}_2\bar{a}_6^* & \bar{a}_2\bar{a}_7^* & \bar{a}_2\bar{a}_8^* & 0\\ 0 & \bar{a}_3\bar{a}_2^* & \left|\bar{a}_3\right|^2 & \bar{a}_3\bar{a}_4^* & 0 & \bar{a}_3\bar{a}_6^* & \bar{a}_3\bar{a}_7^* & \bar{a}_3\bar{a}_8^* & 0\\ 0 & \bar{a}_4\bar{a}_2^* & \bar{a}_4\bar{a}_3^* & \left|\bar{a}_4\right|^2 & 0 & \bar{a}_4\bar{a}_6^* & \bar{a}_4\bar{a}_7^* & \bar{a}_4\bar{a}_8^* & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \bar{a}_6\bar{a}_2^* & \bar{a}_6\bar{a}_3^* & \bar{a}_6\bar{a}_4^* & 0 & \left|\bar{a}_6\right|^2 & \bar{a}_6\bar{a}_7^* & \bar{a}_6\bar{a}_8^* & 0\\ 0 & \bar{a}_7\bar{a}_2^* & \bar{a}_7\bar{a}_3^* & \bar{a}_7\bar{a}_4^* & 0 & \bar{a}_7\bar{a}_6^* & \left|\bar{a}_7\right|^2 & \bar{a}_7\bar{a}_8^* & 0\\ 0 & \bar{a}_8\bar{a}_2^* & \bar{a}_8\bar{a}_3^* & \bar{a}_8\bar{a}_4^* & 0 & \bar{a}_8\bar{a}_6^* & \bar{a}_8\bar{a}_7^* & \left|\bar{a}_8\right|^2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}. \tag{20}$$

The rows and columns of zeros in this matrix are exactly the rows and columns that are affected by the collective dephasing channel, giving rise to the robust character of this density matrix. For analysis in this section, we consider, without loss of generality, the first form of this class $|\psi_1\rangle=\bar{a}_2|2\rangle+\bar{a}_4|4\rangle$, which

allows the decoherence and disentanglement timescales to be seen clearly. If we were to include the entire subclass, we would have the same behavior as our forthcoming analysis.

4.2.1 *Robust Class: Multi-Local Dephasing Channel $\mathcal{EF}$.*

The time-evolved density matrix for the multi-local dephasing channel acting on $|\psi_1\rangle$ is given by

$$\rho_{\rm AB}^{{\rm R},\mathcal{E}F}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0 \\ 0 & |\bar{a}_2|^2 & 0 & \bar{a}_2\bar{a}_4^*\gamma_{\rm A}\gamma_{\rm B} & 0\,0\,0\,0\,0 \\ 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0 \\ 0 & \bar{a}_4\bar{a}_2^*\gamma_{\rm A}\gamma_{\rm B} & 0 & |\bar{a}_4|^2 & 0\,0\,0\,0\,0 \\ 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0 \\ 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0 \\ 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0 \\ 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0 \\ 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0 \end{pmatrix} \tag{21}$$

We see that the off-diagonal elements decay according to the timescale $\tau_{2-{\rm dec},\mathcal{EF}}^{\rm R} = (\frac{1}{\Gamma_1})$. The disentanglement, characterized via the negativity, is given by, $\mathcal{N}\left(\rho_{\rm AB}^{\mathcal{D}}(t)\right) = \frac{\|\rho^{T_{\rm A}}\|_1 - 1}{2} = \frac{\|\rho^{T_{\rm B}}\|_1 - 1}{2} = |\bar{a}_2|\,|\bar{a}_4|\,\gamma_{\rm A}\gamma_{\rm B}$. Disentanglement proceeds at the same single timescale, $\tau_{{\rm dis},\mathcal{EF}}^{\rm R} = (\frac{1}{\Gamma_1})$. The reduced density matrices are given by

$$\rho_{\rm A}^{{\rm R},\mathcal{E}F}(t) = {\rm Tr}_{\rm B}\rho_{\rm AB}^{{\rm R},\mathcal{EF}}(t) = \begin{pmatrix} |\bar{a}_2|^2 & 0 & 0 \\ 0 & |\bar{a}_4|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

$$\rho_{\rm B}^{{\rm R},\mathcal{EF}}(t) = {\rm Tr}_{\rm A}\rho_{\rm AB}^{{\rm R},\mathcal{EF}}(t) \begin{pmatrix} |\bar{a}_4|^2 & 0 & 0 \\ 0 & |\bar{a}_2|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

always fully decohered as in the case of the fragile states. Again, we see that decoherence never proceeds faster than disentanglement, so that clearly the inequality $\tau_{{\rm dis},\mathcal{D}}^{\rm R} \le \tau_{2-{\rm dec},\mathcal{D}}^{\rm R}$ is satisfied.

4.2.2 *Robust Class: Collective Dephasing Channel $\mathcal{D}$.*

Here, $\rho(t) = \rho(0)$ as defined above in Eq. 20. and the state neither decoheres nor disentangles: the off-diagonal elements remain unreduced and the negativity remains at the maximum value of 1 for all times. These states are robust against collective dephasing noise.

4.3 *Conclusions*

We have shown for this model of two-qutrit systems under the chosen basis-specific dephasing noise, whether local, multi-local or collective in nature, that disentanglement proceeds at least as fast as decoherence. This result accords with previous studies of similar nature for qubit systems. Because the case of two-dimensions in quantum mechanics is often a special one in quantum mechanics, for example, in the cases of the foundational theorems of Gleason and Kochen and Specker, it is always important to examine cases beyond that of qubits for the possibility of different behavior from that exhibited in the special case of qubits. Our results suggest that the relation between basis-dependent dephasing decoherence and disentanglement found here can be expected to hold for pairs of initially entangled qu-d-it systems under analogous dephasing noise for general values of d. It remains an open question whether or not this phenomenon can be generalized to an arbitrary *multi-partite* quantum systems, each of arbitrary dimension, however, not least of all because good entanglement measures for mixed states of such systems is lacking.

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